



## Short communication

## A Lagrange-based generalised formulation for the equations of motion of simple walking models

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## ABSTRACT

Simple 2D models of walking often approximate the human body to multi-link dynamic systems, where body segments are represented by rigid links connected by frictionless hinge joints. Performing forward dynamics on the equations of motion (EOM) of these systems can be used to simulate their movement. However, deriving these equations can be time consuming. Using Lagrangian mechanics, a generalised formulation for the EOM of  $n$ -link open-loop chains is derived. This can be used for single support walking models. This has an advantage over Newton–Euler mechanics in that it is independent of coordinate system and prior knowledge of the ground reaction force (GRF) is not required. Alternative strategies, such as optimisation algorithms, can be used to estimate joint activation and simulate motion. The application of Lagrange multipliers, to enforce motion constraints, is used to adapt this general formulation for application to closed-loop chains. This can be used for double support walking models. Finally, inverse dynamics are used to calculate the GRF for these general  $n$ -link chains. The necessary constraint forces to maintain a closed-loop chain, calculated from the Lagrange multipliers, are one solution to the indeterminate problem of GRF distribution in double support models. An example of this method's application is given, whereby an optimiser estimates the joint moments by tracking kinematic data.

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## 1. Introduction

There are numerous examples of researchers using relatively simple dynamic models to investigate the way in which human beings walk (Baker et al., 2004; Buczek et al., 2006; Kuo, 2007; McGrath et al., 2015b; Millard et al., 2011). Some have further expanded to models of 'moderate' complexity (Martin and Schmiedeler, 2014; McGrath et al., 2015a; Pandy and Berme, 1988a, 1988b). Often these latter models consist of a number of rigid links connected by frictionless hinge joints, forming a chain. These represent the segments and joints of a person's limbs. In order for these models to provide forward dynamic simulations of a person's movement, their equations of motion (EOM) must be derived.

General formulae for the EOM of  $n$ -link chains have been previously developed for use in gait modelling, using a Newtonian approach (Pandy and Berme, 1988a). A great advantage of these general formulae is the time saved in developing the EOM for models with a large number of degrees-of-freedom (DOFs), where a

manual approach is very time consuming. This paper describes a similar approach but using Lagrangian mechanics to develop the formulae instead, which are independent of the chosen coordinate frame. Also, because they use energy calculations, rather than forces, prior knowledge of the ground reaction force (GRF) is not required.

Once these equations are developed, walking simulations can be performed using the same methods as the complex models, such as using optimisation to estimate internal kinetics and joint activations (Anderson and Pandy, 2003). This study gives an example of such a simulation.

## 2. Method

## 2.1. Open-loop chains

The Lagrange equation to derive EOM for an open-loop chain is given (Onyshko and Winter, 1980).

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad (1)$$

where  $L$  is the Lagrangian function – the difference between the kinetic and potential energy – and  $q_i$  are the generalised coordinates for the  $i$ th link of the chain.

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Eq. (1) shows the Lagrange equation equal to zero. This is valid when there are no external forces or moments acting on the system. For the derivations outlined here, moments will be acting at the joints between links so the Lagrange equation is adapted.

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i \quad (2)$$

where  $Q_i$  are the generalised forces derived from a consideration of virtual work ( $\delta w$ ):

$$\delta w = \sum_i Q_i \delta q_i \quad (3)$$

Two choices for  $q_i$  are joint angle ( $\varphi_i$ ) or link angle ( $\theta_i$ ) to the vertical.

$$\delta w = \sum_i -M_i \delta \varphi_i = \sum_i M_i (\theta_{i-1} - \theta_i) = \sum_i (M_{i+1} - M_i) \theta_i \quad (4)$$

where  $M_i$  is the moment acting at the distal joint of the  $i$ th link of the chain. This means  $Q_i$  is equal to  $-M_i$  if joint angles are used or  $M_{i+1} - M_i$  if the link angles to the vertical are used. Although selecting the joint angles would decouple the generalised force terms, it makes the functions for the energy calculations more complex. Consequently, link angles to the vertical are preferable and are used throughout this paper.

The following derivation is for an open-loop chain consisting of  $n$  rigid links, where the ground acts as a workless constraint at one end of the chain and the other end is free. Each link has the characteristics shown in Fig. 1. The angular position of the  $i$ th link is defined as the link's angle to the vertical. Anticlockwise is positive for angles and moments. The total length of the link is  $l_i$ . It has a mass,  $m_i$ , acting at a single point, with a moment of inertia,  $I_i$ . The position of the centre-of-mass (CM) of the link is defined by two values,  $d_i$  and  $e_i$ , where  $d_i$  is parallel to the length of the link and  $e_i$  is perpendicular to it. The direction of progression is in the positive  $x$  direction and upwards is the positive  $y$  direction. The acceleration due to gravity is written as  $g$ .

Assumptions are made for these generalised formulae to be valid. There is no branching and each link is connected to adjacent links by frictionless hinge joints. The model is 2D, in the sagittal plane, and the hinge joints are the only DOFs. For each link, there are two controlled muscle moments acting on the proximal and distal ends, respectively.

Firstly, the coordinates of the CMs of each segment are considered:

$$x_i = \sum_{h=1}^{i-1} (-l_h \sin \theta_h) - d_i \sin \theta_i + e_i \cos \theta_i \quad (5)$$

$$y_i = \sum_{h=1}^{i-1} (l_h \cos \theta_h) + d_i \cos \theta_i + e_i \sin \theta_i \quad (6)$$

The linear velocities of these CMs are defined by the first derivatives.

$$\dot{x}_i = \sum_{h=1}^{i-1} (-l_h \cos \theta_h \dot{\theta}_h) - d_i \cos \theta_i \dot{\theta}_i - e_i \sin \theta_i \dot{\theta}_i \quad (7)$$

$$\dot{y}_i = \sum_{h=1}^{i-1} (-l_h \sin \theta_h \dot{\theta}_h) - d_i \sin \theta_i \dot{\theta}_i + e_i \cos \theta_i \dot{\theta}_i \quad (8)$$

The resultant velocities are calculated for each CM.

$$v_i^2 = \dot{x}_i^2 + \dot{y}_i^2 \quad (9)$$

The kinetic energy,  $T$ , and the potential energy,  $V$ , of the system are calculated.

$$T = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 = \sum_{i=1}^n \left( \frac{1}{2} m_i v_i^2 + \frac{1}{2} I_i \dot{\theta}_i^2 \right) \quad (10)$$

$$V = mgh = \sum_{i=1}^n \left( m_i \left( \sum_{h=1}^{i-1} (l_h g \cos \theta_h) + d_i g \cos \theta_i + e_i g \sin \theta_i \right) \right) \quad (11)$$

The Lagrangian function is calculated by subtracting the potential energy from the kinetic.

$$L = T - V \quad (12)$$

Partial differentials of  $L$  with respect to  $\dot{\theta}_i$  and  $\theta_i$  are taken to evaluate the terms in the Lagrangian equation.

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i} = \sum_i (M_{i+1} - M_i) \theta_i \quad (13)$$

From the calculation of these terms, the EOM can be written in matrix form.

$$B \cdot \ddot{\theta} = C \quad \text{where,} \quad \begin{bmatrix} b_{1,1} & \dots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \dots & b_{n,n} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \vdots \\ \ddot{\theta}_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad (14)$$

For a given row,  $p$ , and a given column,  $q$ :

$$b_{p,q} = \begin{cases} \left( m_p d_p^2 + m_p e_p^2 + \left( \sum_{j=p}^n m_{j+1} \right) l_p^2 + I_p \right) & \text{if } p = q \\ \left( m_p d_p + \left( \sum_{j=p}^n m_{j+1} \right) l_p \right) l_q \cos(\theta_q - \theta_p) + (m_p e_p l_q \sin(\theta_p - \theta_q)) & \text{if } p > q \\ \left( m_q d_q + \left( \sum_{j=q}^n m_{j+1} \right) l_q \right) l_p \cos(\theta_p - \theta_q) + (m_q e_q l_p \sin(\theta_q - \theta_p)) & \text{if } q > p \end{cases} \quad (15)$$

$$c_p = \sum_{h=1}^{\{n|p \neq h\}} \left( \dot{\theta}_h^2 \left( \begin{cases} \left( m_p d_p + \sum_{j=p}^n (m_{j+1}) l_p \right) l_h \sin(\theta_h - \theta_p) & \text{if } h < p \\ - \left( m_h d_h + \sum_{j=h}^n (m_{j+1}) l_h \right) l_p \sin(\theta_p - \theta_h) & \text{otherwise} \end{cases} \right) \right) + \left( \begin{cases} (m_p e_p l_h) \cos(\theta_p - \theta_h) & \text{if } h < p \\ - (m_h e_h l_p) \cos(\theta_h - \theta_p) & \text{otherwise} \end{cases} \right) + \left( m_p d_p + \left( \sum_{j=p}^n m_{j+1} \right) l_p \right) g \sin \theta_p - m_p e_p g \cos \theta_p + M_{p+1} - M_p \quad (16)$$

The sigma notation  $\sum_{h=1}^{\{n|p \neq h\}}$  means  $h$  covers all of the values from 1 to  $n$ , but is never the same as  $p$ .

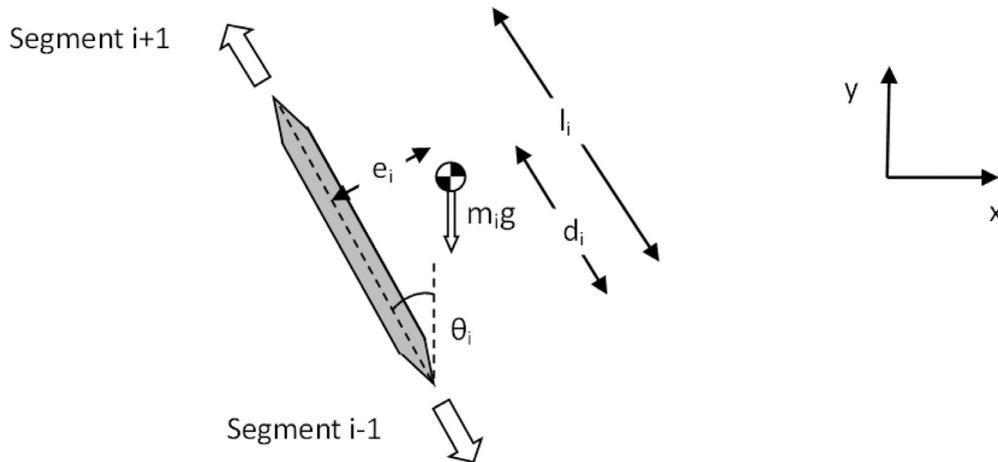


Fig. 1. The notation for describing each segment and its centre-of-mass.

This method does, however, rely on an estimation of joint moments. Later in this study, an optimisation algorithm is described, which uses measured kinematics and estimates these moments. This means that Matrix  $B$  can then be inverted and used to produce the vector  $\ddot{\theta}$ , which gives the angular acceleration for each link of the chain.

## 2.2. Closed-loop chains

Eq. (14) is only applicable for open-loop chains, i.e. single support walking models. To create double support models, closed-loop chains are required. An advantage of Lagrange mechanics is that constraints can be applied relatively simply using 'Lagrange multipliers'.

To apply a constraint, the  $j$ th constraint function ( $f_j$ ) is defined such that:

$$f_j = 0 \quad (17)$$

The governing Lagrange equation is modified to include the Lagrange multipliers:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} - \sum_j \left( \lambda_j \frac{\partial f_j}{\partial q_i} \right) = Q_i \quad (18)$$

where  $\lambda_j$  is the Lagrange multiplier for the  $j$ th constraint. For a number of constraint equations,  $r$ , the same number of new unknown variables need to be solved. This is done by incorporating the constraint equations into the matrix formulation of the EOM, thus solving for  $\ddot{q}_i$  and  $\lambda_j$  simultaneously. If the constraint equations are purely positional (only contain  $q_i$  terms), they need to be differentiated twice so that they contain  $\ddot{q}_i$  terms. This new equation then needs to be separated into two functions; one that contains only the  $\ddot{q}_i$  terms,  $g_i$ , and one that contains the rest of the terms  $h_j$  (Eq. (19)). These terms can now be incorporated into the matrix formulation (Eq. (20)).

$$\frac{d^2 f_j}{dt^2} (\ddot{q}_i, \dot{q}_i, q_i, t) = g_i(\ddot{q}_i, t) + h_j(\dot{q}_i, q_i, t) = 0 \quad (19)$$

$$\begin{bmatrix} b_{ii} & -\frac{\partial f_j}{\partial q_i} \\ g_i(\ddot{q}_i, t) & 0 \end{bmatrix} \begin{bmatrix} \ddot{q}_i \\ \lambda_j \end{bmatrix} = \begin{bmatrix} c_i \\ -h_j(\dot{q}_i, q_i, t) \end{bmatrix} \quad (20)$$

It's important to note that the  $\ddot{q}_i$  terms are no longer all independent. For a chain with  $n$  DOFs and  $r$  constraint equations, only  $n-r$  are independent. If the initial conditions satisfy the constraints, then computing  $\ddot{q}_i$  and integrating to solve for all DOFs should produce solutions which are consistent with the constraint equations. These can be validated using the constraint equations (Ulker, 2010). If  $\ddot{q}_i$  is known for the first  $n-r$  links in the chain, the constraint equations can be used to compute  $\ddot{q}_i$  for the final  $r$  links. A worked example is given in the [appendix](#).

## 2.3. Ground reaction force calculations

Inverse dynamics can be used to calculate the total GRF acting on a walking model. For open-loop chains, this is the GRF where the chain is in contact with the ground (the single supporting foot). For closed-loop chains, a method is required to determine how the total GRF is distributed between the two ground contact points, which is an indeterminate problem. The following derivation is for the vertical and horizontal components of the total GRF.

By considering the vertical direction first, Newton's second law of motion is used:

$$GRF_y - mg = \sum_{i=1}^n m_i \ddot{y}_i \quad (21)$$

Differentiating Eq. (8):

$$\ddot{y}_i = \sum_{h=1}^{i-1} l_h \left( -\ddot{\theta}_h \sin \theta_h - \dot{\theta}_h^2 \cos \theta_h \right) + d_i \left( -\ddot{\theta}_i \sin \theta_i - \dot{\theta}_i^2 \cos \theta_i \right) + e_i \left( \ddot{\theta}_i \cos \theta_i - \dot{\theta}_i^2 \sin \theta_i \right) \quad (22)$$

Similarly, for the horizontal direction:

$$GRF_x = ma = \sum_{i=1}^n m_i \ddot{x}_i \quad (23)$$

Differentiating Eq. (7):

$$\ddot{x}_i = \sum_{h=1}^{i-1} l_h \left( -\ddot{\theta}_h \cos \theta_h + \dot{\theta}_h^2 \sin \theta_h \right) + d_i \left( -\ddot{\theta}_i \cos \theta_i + \dot{\theta}_i^2 \sin \theta_i \right) + e_i \left( -\ddot{\theta}_i \sin \theta_i - \dot{\theta}_i^2 \cos \theta_i \right) \quad (24)$$

During double support, although the total GRF can be calculated, there is an infinite number of ways this can be distributed between the two feet. Ren et al. (2007), solved this problem by making a smooth transition assumption. The Lagrange multipliers method used here offers an alternative approach because the multipliers can be used to calculate the force required to maintain a given constraint. In the case of this study, the forces required to hold the trailing foot fixed to the ground can be used to calculate the GRF under that foot. By using inverse dynamics, in the same way as before, to calculate the total GRF, a simple subtraction can be used to obtain the GRF under the leading foot.

Since the constraint forces are acting upon the trailing foot and it is stationary, it can be assumed that the GRF components beneath it are equal to these constraint forces. The forces the constraints produce can be expressed:

$$F_{q_i} = \lambda \frac{\partial f}{\partial q_i} \quad (25)$$

To calculate the constraint forces in the  $x$  and  $y$  directions, the following equations are used:

$$F_x = \lambda_{f_1} \sum_{i=1}^n \left( \frac{\partial f_1}{\partial \theta_i} \frac{\partial \theta_i}{\partial x} \right) = \lambda_{f_1} \sum_{i=1}^n \left( -l_i \cos \theta_i \cdot \frac{1}{-l_i \cos \theta_i} \right) = \lambda_{f_1} \quad (26)$$

$$F_y = \lambda_{f_2} \sum_{i=1}^n \left( \frac{\partial f_2}{\partial \theta_i} \frac{\partial \theta_i}{\partial y} \right) = \lambda_{f_2} \sum_{i=1}^n \left( -l_i \sin \theta_i \cdot \frac{1}{-l_i \sin \theta_i} \right) = \lambda_{f_2} \quad (27)$$

These values relate to the GRF components at the trailing foot. Subtracting these from their respective total GRF components give the GRF components beneath the leading foot.

## 3. Example simulation

Gait laboratory data was collected for a single, healthy, female participant (28 years old, 65 kg, 162 cm). Ethical approval for the study was granted by the Institutional Ethics Panel (ref HSCR13/18). A Vicon 3D motion capture system (Oxford Metrics plc., Oxford, UK) and Kistler force plates (Kistler Group, Winterthur, Switzerland) were used to capture kinematic and kinetic data, respectively.

The derived generalised formulae were used to generate a seven degree-of-freedom model (previously described by McGrath et al. (2015a)). For the simulation model, the participants anthropometric data were used and segment masses were estimated using Winter's formulae (Winter, 1979, 1991).

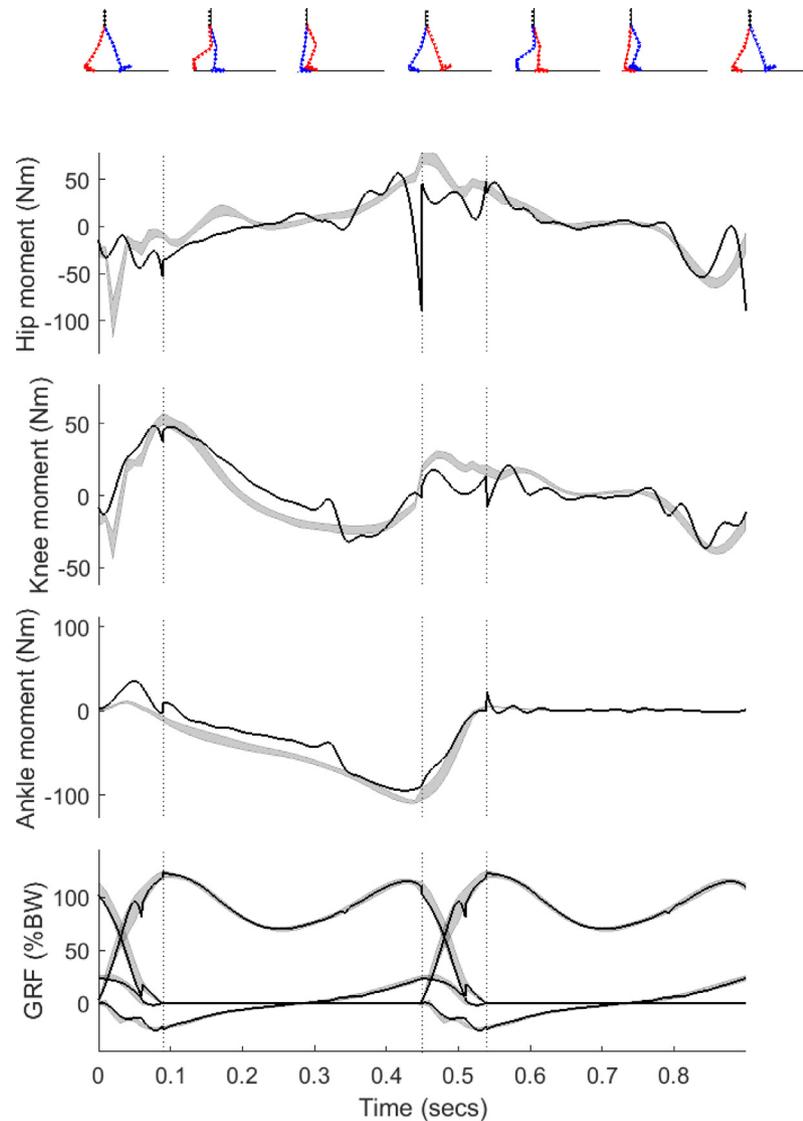
The simulation was split into two: a single support (open chain) and a double support (closed chain). For both double and single support simulations, a global optimisation was performed using the MATLAB function 'GlobalSearch' (Ugray et al., 2007). The input parameters were the initial kinematic state (segment angular positions and velocities) and the joint moments over the whole simulation. The initial kinematic state was known from the gait lab measurements but since the temporal profiles of the joint moments were unknown, the initial estimate was taken from Winter's data (Winter, 1979, 1991). The cost function was the root mean square difference of the predicted kinematics, to those measured in the gait lab. Consequently, the optimiser was designed to 'track' the motion.

The results of this simulation are illustrated in [Fig. 2](#).

## 4. Discussion

A general formulation for the EOM of an open-link chain has been derived and presented here, with the application of modelling bipedal walking. Using Lagrangian mechanics to derive these formulae has been shown to be independent of coordinate frames and requires less prior kinetic knowledge than alternative approaches, such as Newton–Euler mechanics. In terms of walking, this means that the GRF does not need to be known or estimated to perform forward dynamics calculations.

However, joint moments do need to be estimated. This can be executed using an optimisation procedure, a similar method to



**Fig. 2.** The results of the simulation when applied to measured data. The simulation kinematics are shown (solid stick figures) against the equivalent measured kinematics (dotted stick figures). The black lines on the plots show the kinetic outputs of the simulation compared with the equivalent mean  $\pm$  one standard deviation kinetics measured in the gait lab (grey area). The moments shown are for those joints that were on the leading leg at the start of the gait cycle (blue on the stick figures). The ground reaction force (GRF) components are shown in terms of percentage of bodyweight (% BW). The dotted vertical lines indicate transitions between double and single support. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

how Anderson and Pandy (2003) estimated muscle activations in a more complex model with a higher number of degrees-of-freedom. The advantage of simple models is that they allow a focus on the fundamental issues of how the general form and function of the musculoskeletal system affect gait biomechanics without having to be concerned about whether solutions are affected by the precise choice of a number of largely unknown, and in many cases unknowable, secondary parameters such as precise details of the muscle anatomy and physiology. Less complex models will lead to faster solutions within complex iterative optimisation procedures.

The strength of forward dynamic models is their ability to perform 'what-if' simulations. For example, using the described model, to simulate a person with lower knee strength, the knee moment curve could be constrained to illustrate the effect on kinematics. Alternatively, changing the joint angle curves (e.g. a fixed ankle angle) would affect the optimisation process and, ultimately, the moment predictions.

Another advantage of Lagrangian mechanics is that Lagrange multipliers can be incorporated into the calculations to apply constraints. This enables the modelling of a closed-loop chain, which, in terms of walking, equates to the double support phase. Additionally, it has been shown that these multipliers can be used to estimate the distribution of the GRF when both feet are contacting the floor; something that was previously an indeterminate problem.

#### Conflict of interest statement

There are no conflicts of interest related to this work.

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## Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at <http://dx.doi.org/10.1016/j.jbiomech.2017.02.013>.

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